Boundary Backstepping Control of Flow-Induced Vibrations of a Membrane at High Mach Numbers

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We design a controller for flow-induced vibrations of an infinite-band membrane, with flow running across the band and only above it, and with actuation only on the trailing edge of the membrane. Due to the infinite length of the membrane, the dynamics of the membrane in the spanwise direction are neglected, namely, we employ a one-dimensional (1D) model that focuses on streamwise vibrations. This framework is inspired by a flow along an airplane wing with actuation on the trailing edge. The model of the flow-induced vibration is given by a wave partial differential equation (PDE) with an anti-damping term throughout the 1D domain. Such a model is based on linear aeroelastic theory for Mach numbers above 0.8. To design a controller, we introduce a three-stage backstepping transformation. The first stage gets the system to a critically antidamped wave equation, changing the stiffness coefficient's value but not its sign. The second stage changes the system from a critically antidamped to a critically damped equation with an arbitrary damping coefficient. The third stage adjusts stiffness arbitrarily. The controller and backstepping transformation map the original system into a target system given by a wave equation with arbitrary positive damping and stiffness. [DOI: 10.1115/1.4029468]

1 Introduction

Flow-induced vibrations arise in flight and other applications. Many approaches exist in the literature for modeling using aeroelastic theory. Most models are very complicated, with the potential function of the fluid governed by a PDE and coupled, through fluid pressure, with a PDE for the membrane vibrations. This modeling approach is examined in Ref. [1]. It is difficult to design a controller for such a model. Modeling of fluid–structure interaction systems for flutter analysis is pursued in Refs. [2–4]. Most of these studies are geared toward the design of materials' properties for applications.

Flow-induced vibrations can be controlled using actuation either on the boundary of a membrane/plate or in its domain. In Ref. [5], in-domain force actuation is used to control flow-induced vibrations of an elastic plate. Reference [6] studies an energy harvesting system by modifying instead of suppressing vibrations. In Ref. [7], a proportional-integral-derivative controller is employed for an aeroelastic system with a plate.

At large Mach numbers (above 0.8), *piston theory* can be used instead of Bernoulli's equation to define pressure. This approach is widely used in literature [7–11]. When applied to a rectangular membrane with fluid passing over it, this modeling approach results in a flow–structure interaction model given by a wave equation with an *antidamping* term.

Stability analysis for wave PDE systems has been conducted in many references in the literature, including Refs. [12–16]. However, the flow-induced vibrations at high Mach number, despite being modeled by the wave equation, cannot be approached by such analysis techniques because of the presence of the additional antidamping effect.

Control of unstable wave equations has been recently made possible by the development of the backstepping control techniques for boundary control of PDEs [17–21]. Wave PDEs with antidamping have specifically been tackled in Ref. [17] by using adaptive boundary control.

Using the basic idea in Ref. [17], we stabilize the flow-induced vibrations in a 1D model of an infinitely long rectangular membrane with flow running above it and actuation on its trailing edge. Our control design employs a three-stage backstepping transformation. The first stage gets the system to a critically antidamped wave equation, changing the stiffness coefficient's value but not its sign. The second stage changes the system from a critically antidamped to a critically damped equation with an arbitrary damping coefficient. The third stage adjusts stiffness arbitrarily. The controller and backstepping transformation map the original system into a target system given by a wave equation with arbitrary positive damping and stiffness.

This paper is organized as follows: In Sec. 2, we derive a model for our study by using Piston theory [8], the PDE of the membrane vibrations, and two changes of variables. In Sec. 3, we provide a pedagogical exposition of our three-stage backstepping transformation that modifies the systems' damping and stiffness coefficients to desired values, using only boundary control, which is given in Sec. 4. The stability result is given by a theorem in Sec. 5. In Sec. 6, a summary of the three-stage backstepping design is provided along with the composition of the transformations using operators. Simulation results of the system in open and closed-loop are given in Sec. 7.

2 Aeroelastic Model at High Mach Number

We consider an aeroelastic system shown in Fig. 1, with the band $0<x^*<1$, $-\infty<y^*<\infty$, $z^*=0$ representing a membrane. Due to the flow being in the x direction and the membrane being infinite in the y direction, it is assumed that the membrane's vibrations do not depend on y (the vibrations are uniform in y). For the system shown in Fig. 1, by using aeroelastic theory, the PDE of the membrane vibrations is given by

$$w_{tt}(x^*,t)=c^2 w_{x^*x^*}(x^*,t)+p(x^*,z,t)\|_{z^*=0} \tag{1}$$

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where $w(x^*, t)$ is the displacement of the membrane in the z^* direction

$$c = \frac{\tau}{\rho_m} \tag{2}$$

where τ is tension and ρ_m is density of the membrane, and $p(x, z^*, t)\|_{z^*=0}$ is the fluid pressure acting on the membrane.

The pressure is given by

$$p(x^*, z^*, t)\|_{z^*=0} = \frac{\rho U}{\mathrm{M}}\left(\frac{\partial w(x^*, t)}{\partial t} + U\frac{\partial w(x^*, t)}{\partial x^*}\right) \tag{3}$$

where ρ is free-stream density of the fluid, U is velocity, and M is Mach number [8]. Substituting Eq. (3) into Eq. (1), we get

$$\frac{\partial^2 w}{\partial t^2}(x^*, t) = c^2\frac{\partial^2 w}{\partial x^{*2}}(x^*, t) + \mu\frac{\partial w}{\partial t}(x^*, t) + \nu\frac{\partial w}{\partial x^*}(x^*, t) \tag{4}$$

$$w(0, t) = 0 \tag{5}$$

$$w(1, t) = W(t) = \text{Control} \tag{6}$$

where

$$\mu = \frac{\rho U}{\mathrm{M}} \tag{7}$$

$$\nu = U\mu = \frac{\rho U^2}{\mathrm{M}} \tag{8}$$

Before giving the three-stage backstepping transformation, the system (4)–(6) is transformed to a simpler form by using two changes of variables. First, defining

$$v(x^*, t) = w(x^*, t)e^{\frac{\nu}{2c^2}x^*} \tag{9}$$

we eliminate w_{x^*} from the PDE.

Then by using the transformation $x^* \rightarrow cx$, we obtain

$$\frac{\partial^2 v}{\partial t^2}(x, t) = \frac{\partial^2 v}{\partial x^2}(x, t) + 2\lambda\frac{\partial v}{\partial t}(x, t) - \beta v(x, t) \tag{10}$$

$$v(0, t) = 0 \tag{11}$$

$$v\left(\frac{1}{c}, t\right) = V(t) \tag{12}$$

where

$$\lambda = \frac{\rho U}{2\mathrm{M}} \tag{13}$$

$$\beta = \left(\lambda\frac{U}{c}\right)^2 = \frac{\rho^2 U^4}{4\mathrm{M}^2 c^2} \tag{14}$$

and the input is $V(t) = W(t)e^{(\nu/2c^2)}$.

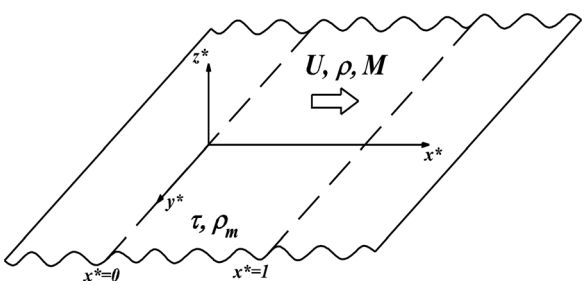


Fig. 1 Infinite rectangular membrane with flow across and above the membrane

3 A Three-Stage Backstepping Transformation

The system (4)–(6) is unstable with positive stiffness and anti-damping. In order to get a stable system, we develop the following three-stage backstepping transformation:

$$w(x^*, t) \mapsto v(x, t) \mapsto \omega(x, t) \mapsto u(x, t) \mapsto \zeta(x, t)$$

with two intermediate target systems and one final target system, as follows:

$$v_{tt} = v_{xx} + 2\lambda v_t - \beta v \qquad \text{anti-damped}$$

$$\omega_{tt} = \omega_{xx} + 2\lambda\omega_t - \lambda^2\omega \qquad \text{critically anti-damped}$$

$$u_{tt} = u_{xx} - 2du_t - d^2 u \qquad \text{critically damped}$$

$$\zeta_{tt} = \zeta_{xx} - 2d\zeta_t - \sigma\zeta \qquad \text{arbitrary damping and stiffness}$$

The states involved in the transformation chain are as follows: $w(x^*, t)$ is the membrane displacement state of the original system; $v(x, t)$ is the exponentially scaled displacement state, $\omega(x, t)$ is the state of the *critically antidamped* intermediate system after the first backstepping transformation; $u(x, t)$ is the state of the *critically damped* intermediate system after the second backstepping transformation; and $\zeta(x, t)$ is the state of the stable target system with *arbitrary damping and stiffness* after the third backstepping transformation.

3.1 The Three Target Systems

3.1.1 First Target Systems. The first intermediate target system is given by

$$\omega_{tt}(x, t) = \omega_{xx}(x, t) + 2\lambda\omega_t(x, t) - \lambda^2\omega(x, t) \tag{15}$$

$$\omega(0, t) = 0 \tag{16}$$

$$\omega\left(\frac{1}{c}, t\right) = \Omega(t) \tag{17}$$

which is a critically antidamped wave equation, with a damping constant 2λ and with a stiffness constant λ^2 instead of β. All eigenvalues of this plant lie on the vertical line [17].

3.1.2 Second Target Systems. The second intermediate target system is given by

$$u_{tt}(x, t) = u_{xx}(x, t) - 2du_t(x, t) - d^2 u(x, t) \tag{18}$$

$$u(0, t) = 0 \tag{19}$$

$$u\left(\frac{1}{c}, t\right) = U(t) \tag{20}$$

which is a critically damped system with a stiffness constant d^2 and a damping constant $2d$. The differences between first and second target systems are the potentially different positive constants λ and d and the sign of the damping coefficient. (Damping is changed from -2λ to $2d$ and stiffness is changed from λ^2 to d^2.)

3.1.3 Third Target Systems. The final stable target system that has general stiffness and damping is given by

$$\zeta_{tt}(x, t) = \zeta_{xx}(x, t) - 2d\zeta_t(x, t) - \sigma\zeta(x, t) \tag{21}$$

$$\zeta(0, t) = 0 \tag{22}$$

$$\zeta\left(\frac{1}{c}, t\right) = 0 \tag{23}$$

The damping constant is not changed but the stiffness is changed from d^2 to σ. The intermediate control inputs $\Omega(t)$ and $U(t)$ are defined in Sec. 4.

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3.2 Three Backstepping Transformations

3.2.1 First Transformation. In order to transform the system (10)–(12) into Eqs. (15)–(17), we use the following transformation:

$$\omega(x,t) = v(x,t) - \int_0^x r(x,y)v(y,t)dy \tag{24}$$

This transformation is only related to displacement.

3.2.2 Second Transformation. The aim of this stage is transforming the critically antidamped system (15)–(17) into the critically damped system (18)–(20) by using the following transformation:

$$u(x,t) = h(x)\omega(x,t) - \int_0^x k(x,y)\omega(y,t)dy - \int_0^x s(x,y)\omega_t(y,t)dy \tag{25}$$

This transformation is related to both the displacement and the velocity.

3.2.3 Third Transformation. The last transformation is similar to the first one. It is related only to displacement. It takes Eqs. (18)–(20) into Eqs. (21)–(23) with the transformation

$$\zeta(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t)dy \tag{26}$$

3.3 Kernels of the Backstepping Transformations

3.3.1 Kernel Function of First Transformation. By using the transformation (24) and the Eqs. (10) and (15), the PDE for $r(x, y)$ is obtained as follows:

$$r_{xx}(x,y) = r_{yy}(x,y) + (\lambda^2 - \beta)r(x,y) \tag{27}$$

$$r(x,x) = -\frac{1}{2}(\lambda^2 - \beta)x \tag{28}$$

$$r(x,0) = 0 \tag{29}$$

where $\lambda^2 - \beta = \rho^2U^2/4Mc^2(c^2 - U^2)$. Noting that $\lambda^2 - \beta > 0$ if and only if $c^2 > U^2$ and noting the properties of the Bessel Functions which are given by [22]

$$I_n(x) = i^{-n}J_n(ix) \tag{30}$$

$$I_n(ix) = i^nJ_n(x) \tag{31}$$

the solution of the PDE (27) is given for $c < U$ by

$$r(x,y) = -(\lambda^2 - \beta)y\frac{I_1\left(\sqrt{(\lambda^2 - \beta)(x^2 - y^2)}\right)}{\sqrt{(\lambda^2 - \beta)(x^2 - y^2)}} \tag{32}$$

where $I_n(x)$ is the Modified Bessel Function of nth order, and for $c < U$ by

$$r(x,y) = (\beta - \lambda^2)y\frac{J_1\left(\sqrt{(\beta - \lambda^2)(x^2 - y^2)}\right)}{\sqrt{(\beta - \lambda^2)(x^2 - y^2)}} \tag{33}$$

where $J_n(x)$ is the Bessel Function of nth order.

3.3.2 Kernel Functions of Second Transformation. Using the transformation (25) with the Eqs. (15) and (18), by making necessary calculations the PDEs for the functions $k(x, y)$, $s(x, y)$ and the function $h(x)$ are given by

$$h(x) = \cosh((\lambda + d)x) \tag{34}$$

$$k_{xx}(x,y) = k_{yy}(x,y) + 2(\lambda + d)s_{yy}(x,y) + (d^2 - \lambda)k(x,y) - 2\lambda^2(\lambda + d)s(x,y) \tag{35}$$

$$k(x,0) = 0 \tag{36}$$

$$k(x,x) = (2\lambda + d)\sinh((\lambda + d)x) \tag{37}$$

$$s_{xx}(x,y) = s_{yy}(x,y) + 2(\lambda + d)k(x,y) + (3\lambda^2 + 2\lambda d + d^2)s(x,y) \tag{38}$$

$$s(x,0) = 0 \tag{39}$$

$$s(x,x) = -\sinh((\lambda + d)x) \tag{40}$$

From Ref. [17], the functions $k(x, y)$ and $s(x, y)$ are given by

$$k(x,y) = (2\lambda + d)\sinh((\lambda + d)y) \tag{41}$$

$$s(x,y) = -\sinh((\lambda + d)y) \tag{42}$$

3.3.3 Kernel Function of Third Transformation. Making similar calculations for Eqs. (26), (18), and (21), the PDE for the kernel function is given by

$$q_{xx}(x,y) = q_{yy}(x,y) + (\sigma - d^2)q(x,y) \tag{43}$$

$$q(x,0) = 0 \tag{44}$$

$$q(x,x) = -\frac{1}{2}(\sigma - d^2)x \tag{45}$$

which has the solution

$$q(x,y) = -(\sigma - d^2)y\frac{I_1\left(\sqrt{(\sigma - d^2)(x^2 - y^2)}\right)}{\sqrt{(\sigma - d^2)(x^2 - y^2)}} \tag{46}$$

for $\sigma > d^2$ and

$$q(x,y) = (d^2 - \sigma)y\frac{J_1\left(\sqrt{(d^2 - \sigma)(x^2 - y^2)}\right)}{\sqrt{(d^2 - \sigma)(x^2 - y^2)}} \tag{47}$$

for $\sigma < d^2$.

3.4 Inverse Backstepping Transformations

3.4.1 Inverse Kernel Function of First Transformation. We can also write the inverse transformation of Eq. (24) as given in Ref. [22] with making similar calculations.

$$v(x,t) = \omega(x,t) + \int_0^x r_i(x,y)\omega(y,t)dy \tag{48}$$

$$r_i(x,y) = -(\lambda^2 - \beta)y\frac{J_1\left(\sqrt{(\lambda^2 - \beta)(x^2 - y^2)}\right)}{\sqrt{(\lambda^2 - \beta)(x^2 - y^2)}} \tag{49}$$

3.4.2 Inverse Kernel Functions of Second Transformation. The inverse transformation between the systems (15) and (10) is given in Ref. [17] as follows:

$$\omega(x,t) = h_i(x)u(x,t) - \int_0^x k_i(x,y)u(y,t)dy - \int_0^x s_i(x,y)u_t(y,t)dy \tag{50}$$

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where the functions $h_i(x)$, $k_i(x, y)$ and $s_i(x, y)$ are solved with similar method using Ref. [17] as

$$h_i(x) = \cosh((\lambda + d)x) \tag{51}$$

$$k_i(x, y) = (2d + \lambda)\sinh((\lambda + d)y) \tag{52}$$

$$s_i(x, y) = \sinh((\lambda + d)y) \tag{53}$$

3.4.3 Inverse Kernel Function of Third Transformation. The inverse transformation of Eq. (46) for the case $\sigma > d^2$ is given

$$u(x, t) = \zeta(x, t) + \int_0^x q_i(x, y)\zeta(y, t)dy \tag{54}$$

$$q_i(x, y) = -(\sigma - d^2)y\frac{J_1\left(\sqrt{(\sigma - d^2)(x^2 - y^2)}\right)}{\sqrt{(\sigma - d^2)(x^2 - y^2)}} \tag{55}$$

in Ref. [22].

4 Control Law

The controllers for target systems and the main systems (4)–(6) are given by using defined functions in this section. Substituting Eq. (32) into Eq. (24), we have the relationships between the

$$\begin{aligned} W(t) = {} & \frac{1}{\cosh((\lambda + d)/c)e^{(\rho U^2/2Mc^2)}} \\ & \times \left[\cosh((\lambda + d)/c)\int_0^{\frac{1}{c}} r\left(\frac{1}{c}, y\right)w(y, t)e^{(\rho U^2/2Mc^2)y}dy\right. \\ & + \int_0^{\frac{1}{c}} \bar{k}\left(\frac{1}{c}, y\right)w(y, t)e^{(\rho U^2/2Mc^2)y}dy \\ & + \int_0^{\frac{1}{c}} \bar{s}\left(\frac{1}{c}, y\right)w_t(y, t)e^{(\rho U^2/2Mc^2)y}dy \\ & + \int_0^{\frac{1}{c}} \left(\int_y^{\frac{1}{c}} \bar{k}\left(\frac{1}{c}, \xi\right)r(\xi, y)d\xi\right)w(y, t)e^{(\rho U^2/2Mc^2)y}dy \\ & \left. + \int_0^{\frac{1}{c}} \left(\int_y^{\frac{1}{c}} \bar{s}\left(\frac{1}{c}, \xi\right)r(\xi, y)d\xi\right)w_t(y, t)e^{(\rho U^2/2Mc^2)y}dy\right] \end{aligned} \tag{56}$$

controllers (17) and (12) as follows:

$$\Omega(t) = V(t) - \int_0^{\frac{1}{c}} r\left(\frac{1}{c}, y\right)v(y, t)dy \tag{57}$$

We define the relationship between the controller of the systems (18)–(20) and (15)–(17) as

$$\begin{aligned} U(t) = {} & h\left(\frac{1}{c}\right)\Omega(t) - \int_0^{\frac{1}{c}} k\left(\frac{1}{c}, y\right)\omega(y, t)dy \\ & - \int_0^{\frac{1}{c}} s\left(\frac{1}{c}, y\right)\omega_t(y, t)dy \end{aligned} \tag{58}$$

Using the transformation (26) and the boundary conditions (20) and (23), the controller for the system (18)–(20) is given by

$$U(t) = \int_0^{\frac{1}{c}} q\left(\frac{1}{c}, y\right)u(y, t)dy \tag{59}$$

Substituting Eq. (59) into Eq. (58), we get

$$\begin{aligned} \Omega(t) = {} & \frac{1}{h\left(\frac{1}{c}\right)}\left[\int_0^{\frac{1}{c}} q\left(\frac{1}{c}, t\right)h(y)\omega(y, t)dy\right. \\ & - \int_0^{\frac{1}{c}} q\left(\frac{1}{c}, t\right)\int_0^y k(y, \xi)\omega(\xi, t)d\xi \\ & - \int_0^{\frac{1}{c}} q\left(\frac{1}{c}, t\right)\int_0^y s(y, \xi)\omega_t(\xi, t)d\xi \\ & + \int_0^{\frac{1}{c}} k\left(\frac{1}{c}, y\right)\omega(y, t)dy \\ & \left. + \int_0^{\frac{1}{c}} s\left(\frac{1}{c}, y\right)\omega_t(y, t)dy\right] \end{aligned} \tag{60}$$

After straightforward calculations, we obtain Eq. (60) as follows:

$$\begin{aligned} \Omega(t) = {} & \frac{1}{h\left(\frac{1}{c}\right)}\int_0^{\frac{1}{c}}\left[q\left(\frac{1}{c}, t\right)h(y) + k\left(\frac{1}{c}, t\right)\right. \\ & \left. - \int_y^{\frac{1}{c}} q\left(\frac{1}{c}, \xi\right)k(\xi, y)d\xi\right]\omega(y, t)dy \\ & + \frac{1}{h\left(\frac{1}{c}\right)}\int_0^{\frac{1}{c}}\left[s\left(\frac{1}{c}, y\right) - \int_y^{\frac{1}{c}} q\left(\frac{1}{c}, \xi\right)s(\xi, y)d\xi\right]\omega_t(y, t)dy \end{aligned} \tag{61}$$

by making necessary integral calculations in Eq. (61), the same function is defined as

$$\begin{aligned} \Omega(t) = {} & \frac{1}{h\left(\frac{1}{c}\right)}\int_0^{\frac{1}{c}} \bar{k}\left(\frac{1}{c}, y\right)\omega(y, t)dy \\ & + \frac{1}{h\left(\frac{1}{c}\right)}\int_0^{\frac{1}{c}} \bar{s}\left(\frac{1}{c}, y\right)\omega_t(y, t)dy \end{aligned} \tag{62}$$

where the functions $\bar{k}(x, y)$ and $\bar{s}(x, y)$ are given by

$$\begin{aligned} \bar{k}(x, y) = {} & (2\lambda + d)I_0\left[\sqrt{(\sigma - d^2)(x^2 - y^2)}\right] \\ & \times \sinh((\lambda + d)y) - (\sigma - d^2)y\frac{I_1\sqrt{(\sigma - d^2)(x^2 - y^2)}}{\sqrt{(\sigma - d^2)(x^2 - y^2)}} \\ & \times \cosh((\lambda + d)y) \end{aligned} \tag{63}$$

$$\bar{s}(x, y) = I_0[\sqrt{(\sigma - d^2)(x^2 - y^2)}]\sinh((\lambda + d)y) \tag{64}$$

Substituting Eq. (62) into Eq. (57), we get

$$\begin{aligned} V(t) = {} & \int_0^{\frac{1}{c}} r\left(\frac{1}{c}, y\right)v(y, t)dy + \frac{1}{h\left(\frac{1}{c}\right)}\left[\int_0^{\frac{1}{c}} \bar{k}\left(\frac{1}{c}, y\right)\omega(y, t)dy\right. \\ & \left. + \int_0^{\frac{1}{c}} \bar{s}\left(\frac{1}{c}, y\right)\omega_t(y, t)dy\right] \end{aligned} \tag{65}$$

Using Eq. (65) together with Eq. (24), we obtain the controller for the systems (10)–(12) as follows:

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$$V(t)=\int_0^{\frac{1}{c}}\left[r\left(\frac{1}{c},y\right)+\frac{\bar{k}\left(\frac{1}{c},y\right)}{\cosh((\lambda+d)/c)}-\int_y^{\frac{1}{c}}\frac{\bar{k}\left(\frac{1}{c},\xi\right)}{\cosh((\lambda+d)/c)}r(\xi,y)d\xi\right]v(y,t)dy\int_0^{\frac{1}{c}}\left[\frac{\bar{s}\left(\frac{1}{c},y\right)}{\cosh((\lambda+d)/c)}-\int_y^{\frac{1}{c}}\frac{\bar{s}\left(\frac{1}{c},\xi\right)}{\cosh((\lambda+d)/c)}r(\xi,y)d\xi\right]v_t(y,t)dy \tag{66}$$

Finally, the relationship between the original system $w(x^*, t)$ and the system obtained with changing of variables $v(x, t)$ is used to define controller, $W(t)$, for the original system (4)–(6). The controller for original systems (4)–(6) is given in Eq. (56) at the bottom of the page. The compact form of Eq. (56) is given by

$$W(t)=\int_0^{\frac{1}{c}}[K_P(y)w(y,t)+K_D(y)w_t(y,t)]dy \tag{67}$$

where

$$K_P(y)=\frac{1}{\cosh((\lambda+d)/c)e^{(\rho U^2/2Mc^2)y}}\times\left[\cosh((\lambda+d)/c)r\left(\frac{1}{c},y\right)+\bar{k}\left(\frac{1}{c},y\right)+\int_y^{\frac{1}{c}}\bar{k}\left(\frac{1}{c},\xi\right)r(\xi,y)d\xi\right] \tag{68}$$

$$K_D(y)=\frac{1}{\cosh((\lambda+d)/c)e^{(\rho U^2/2Mc^2)y}}\times\left[\bar{s}\left(\frac{1}{c},y\right)+\int_y^{\frac{1}{c}}\bar{s}\left(\frac{1}{c},\xi\right)r(\xi,y)d\xi\right] \tag{69}$$

5 Stability Result

Let $H_L^1(0,1)$ denote the space

$$H_L^1(0,1):=w\in H^1(0,1);w(0)=0$$

THEOREM 1. *For any initial condition* $(w(\cdot,0),w_t(\cdot,0))\in H_L^1(0,1)\times L^2(0,1)$ *that is compatible with the feedback law* (56), *there exists a unique solution of the closed-loop system* (4)–(6) and (56) *in the space* $C([0,\infty);H_L^1(0,1))\cap C^1([0,\infty);L^2(0,1))$. *Moreover, there exist positive constants* C and γ *independent of the initial data such that the solutions satisfy*

$$\|(w(\cdot,t),w_t(\cdot,t))\|_{H^1(0,1)\times L^2(0,1)}\le Ce^{-\gamma t}\|(w(\cdot,0),w_t(\cdot,0))\|_{H^1(0,1)\times L^2(0,1)} \tag{70}$$

6 Summary: Composition of the Three Backstepping Transformations

In this section, the three-stage backstepping transformation is presented in an alternative fashion, using matrix transformations. This section should shed more insight into the transformation process for a more mathematically inclined reader who wants to see clearly the sequence of three transformations of the full state of the wave equation system, which consists of both displacement and velocity at each of the three stages.

The composition of the three transformations, $v\mapsto\omega\mapsto u\mapsto\zeta$ is given by

$$\begin{aligned}\zeta(x,t)&=\cosh((\lambda+d)x)v(x,t)-\cosh((\lambda+d)x)\int_0^x r(x,y)v(y,t)dy\\&-\int_0^x\bar{k}(x,y)v(y,t)dy-\int_0^x\bar{s}(x,y)v_t(y,t)dy\\&-\int_0^x\left(\int_y^x\bar{k}(x,\xi)r(\xi,y)d\xi\right)v(y,t)dy\\&-\int_0^x\left(\int_y^x\bar{s}(x,\xi)r(\xi,y)d\xi\right)v_t(y,t)dy\end{aligned} \tag{71}$$

However, since the full state of a wave equation is composed of both the displacement and velocity, the sequence of transformations that accounts for the full PDE states at each stage should be viewed as

$$\begin{bmatrix}v(x,t)\\v_t(x,t)\end{bmatrix}\xrightarrow{\mathbb{T}_1}\begin{bmatrix}\omega(x,t)\\\omega_t(x,t)\end{bmatrix}\xrightarrow{\mathbb{T}_2}\begin{bmatrix}u(x,t)\\u_t(x,t)\end{bmatrix}\xrightarrow{\mathbb{T}_3}\begin{bmatrix}\zeta(x,t)\\\zeta_t(x,t)\end{bmatrix} \tag{72}$$

The following figure shows these transformations and the relationships among the target systems:

$$\begin{array}{ccccccc}v(x,t) & \longrightarrow & \omega(x,t) & \longrightarrow & u(x,t) & \longrightarrow & \zeta(x,t)\\ & & & \searrow\ \nearrow & & & \\ v_t(x,t) & \longrightarrow & \omega_t(x,t) & \longrightarrow & u_t(x,t) & \longrightarrow & \zeta_t(x,t)\end{array}$$

The definitions of the functions $\omega_t(x, t)$ and $u_t(x, t)$ are needed to derive the relations between the functions shown above. Recall the transformation (25) to find an alternative form of it. Subtracting Eq. (15) from the differentiated version of Eq. (25) in time, the alternative form is obtained after two integrations by parts as follows:

$$\begin{aligned}u_t(x,t)&=h(x)\omega_t(x,t)-s(x,x)\omega_x(x,t)+s_y(x,x)\omega(x,t)\\&-\int_0^x[k(x,y)+2\lambda s(x,y)]\omega_t(y,t)dy\\&-\int_0^x[s_{yy}(x,y)-\lambda^2 s(x,y)]\omega(y,t)dy\end{aligned} \tag{73}$$

By using the defined functions, this equation is also given by

$$\begin{aligned}u_t(x,t)&=\cosh((\lambda+d_2)x)\omega_t(x,t)\\&+\sinh((\lambda+d_2)x)\omega_x(x,t)\\&-(\lambda+d_2)\sinh(\lambda+d_2)x)\omega(x,t)\\&-\int_0^x d_2\sinh((\lambda+d_2)y)\omega_t(y,t)dy\\&+\int_0^x[(\lambda+d_2)^2+\lambda^2]\sinh((\lambda+d_2)y)\omega(y,t)dy\end{aligned} \tag{74}$$

Similarly, after differentiating in time Eq. (50), by using Eq. (18) and performing integration by parts two times, ω_t is given by

$$\begin{aligned}\omega_t(x,t)&=h_i(x)u_t(x,t)-s_i(x,x)u_x(x,t)\\&+s_{iy}(x,x)u(x,t)\\&-\int_0^x[k_i(x,y)-2d_2s_i(x,y)]u_t(y,t)dy\\&-\int_0^x[s_{iyy}(x,y)-d_2^2s_i(x,y)]u(y,t)dy\end{aligned} \tag{75}$$

Next, we define the 2×2 matrix transformations $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}_3$. The first one among them is given by

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$$\begin{bmatrix} \omega(x,t) \\ \omega_t(x,t) \end{bmatrix} = \mathbb{T}_1 \begin{bmatrix} v(x,t) \\ v_t(x,t) \end{bmatrix} \tag{76}$$

$$\mathbb{T}_1 = \begin{bmatrix} \mathrm{Id} - (K_r \bullet)(x) & 0 \\ 0 & \mathrm{Id} - (K_r \bullet)(x) \end{bmatrix} \tag{77}$$

where by using the Hilbert–Schmidt notation $(K_r \bullet)(x) = \int_0^x r(x,y) \bullet (y) dy$. The second transformation matrix $\mathbb{T}_2$ is given by

$$\begin{bmatrix} u(x,t) \\ u_t(x,t) \end{bmatrix} = \mathbb{T}_2 \begin{bmatrix} \omega(x,t) \\ \omega_t(x,t) \end{bmatrix} \tag{78}$$

$$\mathbb{T}_2 = \begin{bmatrix} h(x) - (K_k \bullet)(x) & -(K_s \bullet)(x) \\ -s(x,x)\partial_x & h(x) \\ +s_y(x,x) - (K_{ss} \bullet)(x) & -(K_{ks} \bullet)(x) \end{bmatrix} \tag{79}$$

where the operators

$$(K_k \bullet)(x) = \int_0^x k(x,y) \bullet (y) dy \tag{80}$$

$$(K_s \bullet)(x) = \int_0^x s(x,y) \bullet (y) dy \tag{81}$$

$$(K_{ks} \bullet)(x) = \int_0^x [k(x,y) + 2\lambda s(x,y)] \bullet (y) dy \tag{82}$$

$$(K_{ss} \bullet)(x) = \int_0^x [s_{yy}(x,y) - \lambda^2 s(x,y)] \bullet (y) dy \tag{83}$$

The third matrix transformation is given by

$$\begin{bmatrix} \zeta(x,t) \\ \zeta_t(x,t) \end{bmatrix} = \mathbb{T}_3 \begin{bmatrix} u(x,t) \\ u_t(x,t) \end{bmatrix} \tag{84}$$

$$\mathbb{T}_3 = \begin{bmatrix} \mathrm{Id} - (K_q \bullet)(x) & 0 \\ 0 & \mathrm{Id} - (K_q \bullet)(x) \end{bmatrix} \tag{85}$$

where $(K_q \bullet)(x) = \int_0^x q(x,y) \bullet (y) dy$. The relationship of the last target systems (21)–(23) and the systems (10)–(12) is given by

$$\begin{bmatrix} \zeta(x,t) \\ \zeta_t(x,t) \end{bmatrix} = \mathbb{T}_3 \mathbb{T}_2 \mathbb{T}_1 \begin{bmatrix} v(x,t) \\ v_t(x,t) \end{bmatrix} \tag{86}$$

$$\mathbb{T}_3 \mathbb{T}_2 \mathbb{T}_1 = \begin{bmatrix} h(x) - (K_k \bullet)(x) & \\ -(K_q(h(x)\bullet))(x) - (K_q(K_k \bullet)(x))(x) & -(K_s \bullet)(x) - (K_q((K_s \bullet)(x)))(x) \\ -h(x)(K_r \bullet)(x) + (K_k((K_r \bullet)(x)))(x) & +(K_s((K_r \bullet)(x)))(x) \\ +(K_q(h(x)(K_r \bullet)(x)))(x) & +(K_q(K_s(K_r \bullet (x))(x)))(x) \\ +(K_q(K_k(K_r \bullet)(x))(x))(x) & \\ s(x,x)\partial_x - s_y(x,x) - (K_{ss} \bullet)(x) & h(x) - (K_{ks} \bullet)(x) - (K_q(h(x)\bullet)(x) \\ +(K_q(s(x,x)\partial_x \bullet))(x) + (K_q(K_{ss} \bullet)(x))(x) & +(K_q((K_{ks} \bullet)(x)))(x) - h(x)(K_r \bullet)(x) \\ +(K_{ss}((K_r \bullet)(x)))(x) - (K_q(s_{yy}(x,x)(K_r \bullet)(x))(x) & +(K_{ks}((K_r \bullet)(x)))(x) \\ +s(x,x)\partial_x(K_r \bullet)(x) - (K_q(s(x,x)\partial_x((K_r \bullet)(x))))(x) & +(K_q(h(x)(K_r \bullet)(x))(x) \\ +s_{yy}(x,x)(K_r \bullet)(x) - (K_q(K_{ss}(K_r \bullet)(x)))(x))(x) & -(K_q(K_{ks}(K_r \bullet)(x))(x))(x) \end{bmatrix} \tag{87}$$

where the detailed form is given by Eq. (71) and by

$$\begin{aligned} \zeta_t(x,t) &= \cosh((\lambda + d)x) v_t(x,t) \\ &\quad - \cosh((\lambda + d)x) \int_0^x r(x,y) v_t(y,t) dy \\ &\quad - \int_0^x \bar{k}(x,y) v_t(y,t) dy - \int_0^x 2\lambda \bar{s}(x,y) v_t(y,t) dy \\ &\quad + \int_0^x \beta \bar{s}(x,y) v(y,t) dy + \bar{s}_y(x,x) v(x,t) \\ &\quad - \int_0^x \bar{s}_{yy}(x,y) v(y,t) dy - \bar{s}(x,x) v_x(x,t) \\ &\quad - \int_0^x \left(\int_y^x \bar{k}(x,\xi) r(\xi,y) d\xi \right) v_t(y,t) dy \\ &\quad - \int_0^x 2\lambda \left(\int_y^x \bar{s}(x,\xi) r(\xi,y) d\xi \right) v_t(y,t) dy \\ &\quad + \int_0^x \beta \left(\int_y^x \bar{s}(x,\xi) r(\xi,y) d\xi \right) v(y,t) dy \\ &\quad - \int_0^x \left(\int_y^x \bar{s}(x,\xi) r_{yy}(\xi,y) d\xi \right) v(y,t) dy \end{aligned} \tag{88}$$

The composition of the matrix transformations $\mathbb{T}_3 \mathbb{T}_2 \mathbb{T}_1$ is shown in the full 2×2 form in Eq. (87). The inverse matrix transformation are given by

$$\begin{bmatrix} v(x,t) \\ v_t(x,t) \end{bmatrix} = \mathbb{T}_1^{-1} \begin{bmatrix} \omega(x,t) \\ \omega_t(x,t) \end{bmatrix} \tag{89}$$

$$\begin{bmatrix} \omega(x,t) \\ \omega_t(x,t) \end{bmatrix} = \mathbb{T}_2^{-1} \begin{bmatrix} u(x,t) \\ u_t(x,t) \end{bmatrix} \tag{90}$$

$$\begin{bmatrix} u(x,t) \\ u_t(x,t) \end{bmatrix} = \mathbb{T}_3^{-1} \begin{bmatrix} \zeta(x,t) \\ \zeta_t(x,t) \end{bmatrix} \tag{91}$$

$$\begin{bmatrix} v(x,t) \\ v_t(x,t) \end{bmatrix} = \mathbb{T}_1^{-1} \mathbb{T}_2^{-1} \mathbb{T}_3^{-1} \begin{bmatrix} \zeta(x,t) \\ \zeta_t(x,t) \end{bmatrix} \tag{92}$$

where

$$\mathbb{T}_1^{-1} = \begin{bmatrix} \mathrm{Id} + (K_{ri} \bullet)(x) & 0 \\ 0 & \mathrm{Id} + (K_{ri} \bullet)(x) \end{bmatrix} \tag{93}$$

$$\mathbb{T}_2^{-1} = \begin{bmatrix} h_i(x) - (K_{ki} \bullet)(x) & -(K_{si} \bullet)(x) \\ -s_i(x,x)\partial_x & h_i(x) \\ +s_{iy}(x,x) - (K_{ssi} \bullet)(x) & -(K_{ksi} \bullet)(x) \end{bmatrix} \tag{94}$$

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$$\mathbb{T}_3^{-1} = \begin{bmatrix} \mathrm{Id} + (K_{qi}\bullet)(x) & 0 \\ 0 & \mathrm{Id} + (K_{qi}\bullet)(x) \end{bmatrix} \tag{95}$$

and

$$(K_{ri}\bullet)(x) = \int_0^x r_i(x,y) \bullet (y)dy \tag{96}$$

$$(K_{ki}\bullet)(x) = \int_0^x k_i(x,y) \bullet (y)dy \tag{97}$$

$$(K_{si}\bullet)(x) = \int_0^x s_i(x,y) \bullet (y)dy \tag{98}$$

$$(K_{ksi}\bullet)(x) = \int_0^x [k_i(x,y) - 2d_2 s_i(x,y)] \bullet (y)dy \tag{99}$$

$$(K_{ssi}\bullet)(x) = \int_0^x [s_{iyy}(x,y) - d_2^2 s(x,y)] \bullet (y)dy \tag{100}$$

$$(K_{qi}\bullet)(x) = \int_0^x q_i(x,y) \bullet (y)dy \tag{101}$$

The composition of the matrix transformations $\mathbb{T}_3^{-1}\mathbb{T}_2^{-1}\mathbb{T}_1^{-1}$ is shown in the full 2×2 form in Eq. (103). The relationship between the boundary conditions (12) and (23) is given by

$$\begin{bmatrix} \zeta \\ \zeta_t \end{bmatrix}\left(\frac{1}{c}, t\right) = \mathbb{T}_3\mathbb{T}_2\mathbb{T}_1 \begin{bmatrix} v \\ v_t \end{bmatrix}\left(\frac{1}{c}, t\right) \tag{102}$$

$$\mathbb{T}_1^{-1}\mathbb{T}_2^{-1}\mathbb{T}_3^{-1} = \begin{bmatrix} h_i(x) - (K_{ki}\bullet)(x) + (K_{ri}(h_i(x)\bullet))(x) & -(K_{si}\bullet)(x) - (K_{ri}(K_{si}\bullet)(x))(x) \\ -(K_{ri}(K_{ki}\bullet)(x))(x) + h_i(x)(K_{qi}\bullet)(x) & -(K_{si}(K_{qi}\bullet)(x))(x) \\ -(K_{ki}(K_{qi}\bullet)(x))(x) + (K_{ki}h_i(x)(K_{qi}\bullet)(x))(x) & -(K_{ri}(K_{ki}(K_{qi}\bullet)(x))(x))(x) \\ -(K_{ri}(K_{si}(K_{qi}\bullet)(x))(x))(x) & \\ & \\ -s_i(x,x)\partial_x + s_{iy}(x,x) - (K_{ssi}\bullet)(x) & h_i(x) - (K_{ksi}\bullet)(x) + (K_{ri}(h_i(x)\bullet))(x) \\ +(K_{ri}(s_i(x,x)\partial_x\bullet))(x) + (K_{ri}(s_{iy}(x,x)\bullet))(x) & -(K_{ri}(K_{ksi}\bullet)(x))(x) \\ -(K_{ri}(K_{ssi}\bullet)(x))(x) - s_i(x,x)\partial_x(K_{qi}\bullet)(x) & h_i(x)(K_{qi}\bullet)(x) - (K_{ksi}(K_{qi}\bullet)(x))(x) \\ +s_{iy}(x,x)(K_{qi}\bullet)(x) - (K_{ssi}(K_{qi}\bullet)(x))(x) & +(K_{ri}(h_i(x)(K_{qi}\bullet)(x)))(x) \\ +(K_{ri}(s_i(x,x)\partial_x(K_{qi}\bullet)(x)))(x) & -(K_{ri}(K_{ksi}(K_{qi}\bullet)(x))(x))(x) \\ +(K_{ri}(s_{iy}(x,x)(K_{qi}\bullet)(x)))(x) & \\ -(K_{ri}(K_{ssi}(K_{qi}\bullet)(x))(x))(x) & \end{bmatrix} \tag{103}$$

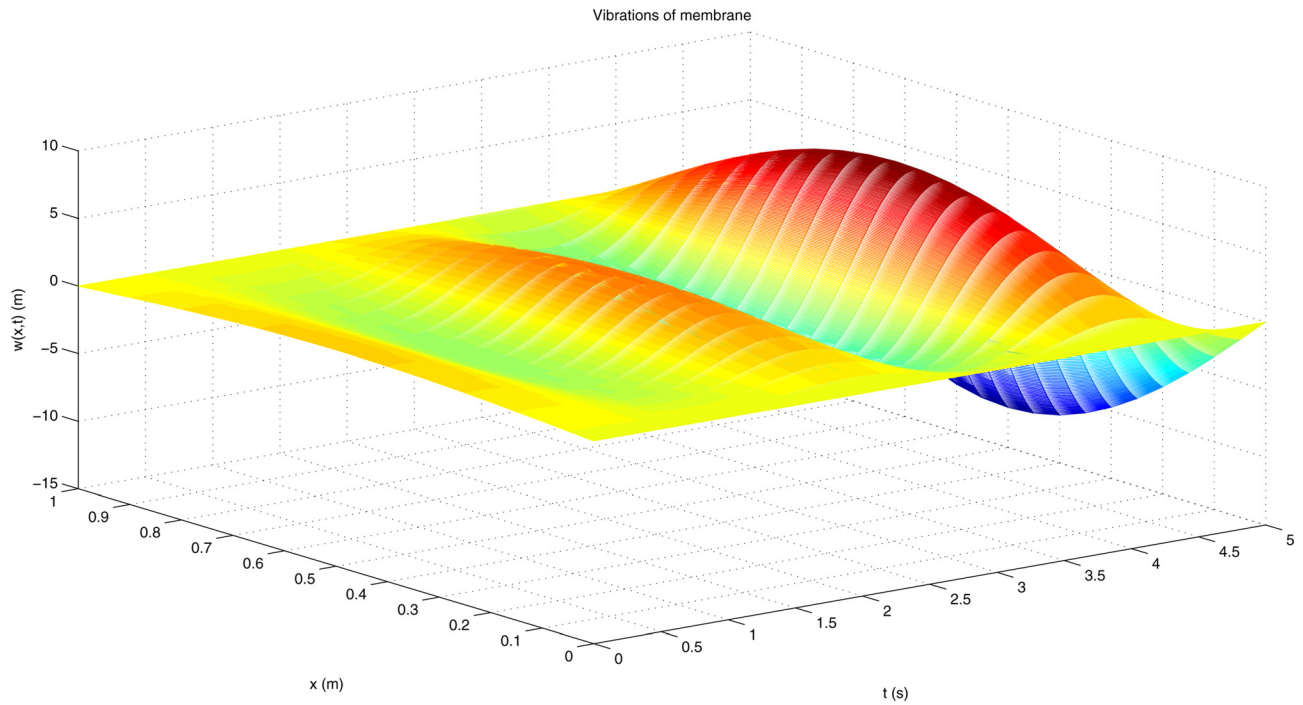


Fig. 2 System response for the systems (10)–(12) without control

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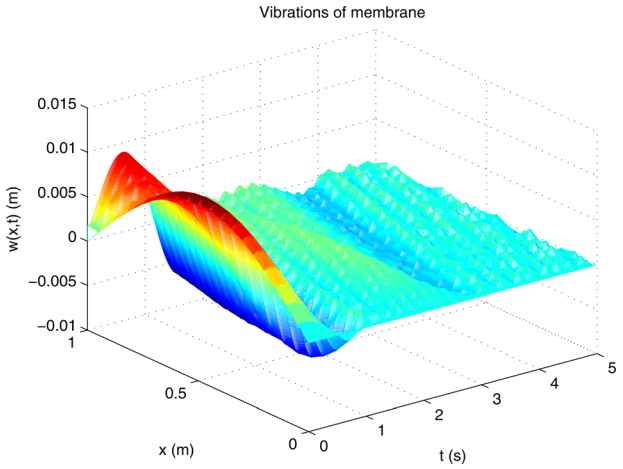


Fig. 3 System response for the systems (10)–(12) with control

7 Simulation

Figure 2 shows that open-loop system is unstable, whereas Fig. 3 demonstrates that the closed-loop systems (10)–(12) is stabilized by our boundary controller. The parameters used in the system are $\lambda = 0.5$, $\beta = \lambda^2$, $\sigma = 5$, and $d = 1$.

References

[1] Uzal, E., and Kapkin, S., 2010, "Vibrations of an Infinite Plate Placed in a Circular Channel Containing Fluid Flow," Aircr. Eng. Aerosp. Technol., **81**(6), pp. 533–535.

[2] Vedeneev, V. V., 2012, "Vibrations of an Infinite Plate Placed in a Circular Channel Containing Fluid Flow," J. Fluids Struct., **29**, pp. 79–96.

[3] Doare, O., Sauzade, M., and Eloy, C., 2011, "Flutter of an Elastic Plate in a Channel Flow: Confinement and Finite-Size Effects," J. Fluids Struct., **27**(1), pp. 76–88.

[4] Zhang, W. W., Ye, Z. Y., and Zhang, C. A., 2009, "Supersonic Flutter Analysis Based on a Local Piston Theory," AIAA J., **47**(10), pp. 2321–2328.

[5] Uzal, E., and Korbahti, B., 2010, "Vibration Control of an Elastic Strip by a Singular Force," Sadhana, **35**(2), pp. 233–240.

[6] Singh, K., Michelin, S., and Langre, E., 2012, "Energy Harvesting From Axial Fluid-Elastic Instabilities of a Cylinder," J. Fluids Struct., **30**, pp. 159–172.

[7] Korbahti, B., 2010, "Specially Orthotropic Panel Flutter Control Using PID Controller," Acta Mech., **212**(3–4), pp. 191–197.

[8] Dowell, E. H., and Hall, K. C., 2001, "Modeling of Fluid-Structure Iteration," Annu. Rev. Fluid Mech., **33**, pp. 445–490.

[9] Kumhaar, H., 1963, "The Accuracy of Applying Linear Piston Theory to Cylindrical Shells," AIAA Journal, pp. 1448–1449.

[10] Dowell, E. H., Crawley, E. F., Curtiss, H. C., Peters, D. A., Scanlan, R. H., and Sisto, F., 1995, *A Modern Course in Aeroelasticity*, Kluwer Academic Publisher, Dordrecht/Boston.

[11] Epureanu, B. I., and Yin, S. H., 2004, "Identification of Damage in an Aeroelastic System Based on Attractor Deformations," Comput. Struct., **82**(31–32), pp. 2743–2751.

[12] Freitas, P., and Zuazua, E., 1996, "Stability Results for the Wave Equation With Indefinite Damping," J. Differ. Equations, **132**(2), pp. 338–352.

[13] Menz, G., 2007, "Exponential Stability of Wave Equations With Potential and Indefinite Damping," J. Differ. Equations, **242**(1), pp. 171–191.

[14] Rivera, J. E. M., and Racke, R., 2008, "Exponential Stability for Wave Equations With Non-Dissipative Damping," Nonlinear Anal., **68**(9), pp. 2531–2551.

[15] Cox, S., and Zuazua, E., 1994, "The Rate at Which Energy Decays in a String Damped at One End," Commun. Partial Differ. Equ., **19**(1–2), pp. 213–243.

[16] Tebou, L., 2007, "Stabilization of the Elastodynamic Equations With a Degenerate Locally Distributed Dissipation," Syst. Control Lett., **56**(7–8), pp. 538–545.

[17] Smyshlyaev, A., Cerpa, E., and Krstic, M., 2010, "Boundary Stabilization of a 1-D Wave Equation With In-Domain Antidamping," SIAM J. Control Optim., **48**(6), pp. 4014–4031.

[18] Krstic, M., 2011, "Dead-Time Compensation for Wave/String PDEs," ASME J. Dyn. Syst. Meas. Control, **133**(3), p. 031004.

[19] Krstic, M., Guo, B. Z., Balogh, A., and Smyshlyaev, A., 2008, "Output-Feedback Stabilization of an Unstable Wave Equation," Automatica, **44**(1), pp. 63–74.

[20] Krstic, M., and Smyshlyaev, A., 2008, "Adaptive Control of PDEs," Annu. Rev. Control, **32**(2), pp. 149–160.

[21] Krstic, M., 2009, "American Control Conference," (ACC'09), St. Louis, MO, June 10–12, pp. 1505–1510.

[22] Krstic, M., and Smyshlyaev, A., 2008, *Boundary Control of PDEs: A Course on Backstepping Designs*, SIAM, Philadelphia.

[23] Brake, M. R., and Segalman, D. J., 2010, "Nonlinear Model Reduction of von Kármán Plates Under Quasi-Steady Fluid Flow," AIAA J., **48**(10), pp. 2339–2347.

[24] Queiroz, M. S., and Rahn, C. D., 2002, "Boundary Control of Vibration and Noise in Distributed Parameter Systems: An Overview," Mech. Syst. Signal Process., **16**(1), pp. 19–38.

[25] Knight, J. J., Lucey, A. D., and Shaw, C. T., 2010, "Fluid–Structure Interaction of a Two–Dimensional Membrane in a Flow With a Pressure Gradient With Application to Convertible Car Roofs," J. Wind Eng. Ind. Aerodyn., **98**(2), pp. 65–72.

[26] Lucey, A. D., Cafolla, G. J., Carpenter, P. W., and Yang, M., 1997, "The Nonlinear Hydroelastic Behaviour of Flexible Walls," J. Fluids Struct., **11**(7), pp. 717–744.

[27] Hansen, S., 2001, "Exact Controllability of an Elastic Membrane Coupled With a Potential Fluid," Int. J. Appl. Math. Comput. Sci., **11**(6), pp. 1231–1248.

[28] Yang, F., and Yao, R., 1996, "The Solution for Mixed Boundary Value Problems of Two-Dimensional Potential Theory," Indian J. Pure Appl. Math., **27**(3), pp. 313–322.

[29] Smyshlyaev, A., and Krstic, M., 2005, "Backstepping Observers for a Class of Parabolic PDEs," Syst. Control Lett., **54**(7), pp. 613–625.

[30] Krstic, M., Kanellakopoulos, I., and Kokotovic, P., 1995, *Nonlinear and Adaptive Control Design*, Wiley, New York.

[31] Serrani, A., and Isidori, A., 2000, "Global Robust Output Regulation for a Class of Nonlinear Systems," Syst. Control Lett., **39**(2), pp. 133–139.

[32] Bisplinghoff, R. L., and Ashley, H., 1962, *Principals of Aeroelasticity*, Dover Publication, New York.

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